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A SPLINE-BASED STATE RECONSTRUCTOR FOR ACTIVE VIBRATION CONTROL OF A FLEXIBLE BEAM

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In this paper a technique for "instantaneous" reconstruction of state variables of a flexible beam is proposed. The reconstructor uses spline shape functions to interpolate the available measurements and to take into account the boundary conditions. Unlike usual dynamic observers, this technique does not require a copy of the system neither any information about the inputs hence it is able to work even in the presence of persistent disturbances and uncertain parameters. The spline functions introduce a sort of "spatial filtering" on the high-frequency modes, this phenomenon increases the robustness of the control scheme against spillover. Computer simulations show that this reconstructor, joined to a suitable controller, is able to reduce the vibration of beams subject to persistent multifrequency disturbances acting at unknown beam abscissae.

1. INTRODUCTION

In the last few years there has been increasing interest around vibration suppression of flexible structures, in particular, those research dealing with the problem of reducing the noise generated by flexible structures when they are excited by some external pseudo-periodic causes [1].

The vibration motion in flexible systems is described by means of partial differential equations and, except for some simple cases, no closed-form solution can be expected. To overcome this difficulty the partial differential equations are usually replaced, via spatial discretization, by means of a finite set of ordinary differential equations. Implicit in this approach is a system truncation: a system of infinite order is replaced by a finite order one [2, 3].

Unfortunately, the controller designed on the finite dimensional approximation, may destabilize the real system [4, 5]. This phenomenon was firstly investigated by Balas in [6] and was termed "spillover": the measurements of the sensors contain both modelled and unmodelled dynamics, hence, when they are feedback, they may persistently excite the unmodelled dynamics and, then, drive the system to instability.

This phenomenon may be suppressed by avoiding the interaction between modelled and unmodelled dynamics. In order to reduce this interaction various approaches have been proposed. In reference [7] an observer is designed on the basis of a system model of order greater than that used to design the controller, in reference [6] the sensor data are prefiltered with a comb filter, in reference [8] a delayed resonator is used, in references [9, 10] it is shown that a control system based on distributed actuators and distributed

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sensors is not affected by spillover phenomenon (for further approaches see references [2, 3] and the references therein).

The colocated control [11–13] is unaffected by spillover phenomenon. In this control technique the structure is forced by means of n signals proportional to the measurements in the same points. In other words the control is constituted by n parallel SISO schemes. To increase performances, MIMO techniques must be used, namely state feedback techniques. Unfortunately, the presence of a dynamic observer needed for state feedback controllers, may exaggerate the spillover phenomenon.

In this work the state variables, needed for state feedback controllers, are reconstructed by means of spline functions. The idea to use spline functions to reconstruct flexible system variables has been proposed also in reference [14] where a set of experimental results are used to determinate the spline coefficients.

In this paper the spline functions are designed in order to interpolate the available measurements and to take into account the boundary conditions. In particular, a spline function is defined for each class of physical homogeneous state variables (e.g. displacement, velocity of deformation, etc.).

This reconstruction is not based on the mathematical model of the structure neither it uses information about inputs, then it is insensible to parameter uncertainties and it may work even in the presence of persistent unknown disturbances. However the most interesting property is that it operates as a spatial filter, i.e. it represents the spatial deformation by means of the smoothest spatial modes, and this contributes to reduce the spillover phenomenon.

Unfortunately, this is also the main drawback of this reconstructor. Indeed it is able to correctly estimate only the first v modes of the structure, where v depends on the sensors number and location. Hence the controller too must depend exclusively on these modes. This kind of controller can be designed through the LQ technique with a proper choice of weighting matrices.

Some preliminary and interesting results on this spline reconstructor have been presented [15, 16].

This paper is organized as follows. In section 2, the flexible beam model used for the controller design is described. In section 3 the spline reconstructor is presented, whereas in section 4 the controller is developed. In section 5 simulation results and robustness analyses are collected. Finally, in section 6 some conclusive remarks are presented.

2. MODEL OF A FLEXIBLE BEAM

This paper is focused on transverse vibration of a beam like structure. For this class of system the free motion is described by the well known equation:

$$\mathscr{M}\ddot{v}(x, t) + EIv^{(4)}(x, t) = 0 \tag{1}$$

where $\mathscr{M}$ is the mass per unit of length, $v(x, t)$ is the vertical displacement, EI is the flexural rigidity. A dot indicates a partial derivative with respect to the time variable t and the superscript in a parentheses represents the derivative order with respect to the spatial variable x. In the following, two different situations have been considered: a simply supported beam and a cantilever beam. In these cases the boundary conditions are given in Table 1 where L is the length of the beam.

In order to obtain a finite dimensional model, the flexible beam is divided into n elements, and the shape of each of them is described, at each time instant, by the corresponding elastic strain.

TABLE 1

Displacement boundary conditions

Simple supported	Cantilevered	
$v(x, t) = 0$ $v^{(2)}(x, t) = 0$	$v(x, t) = 0$ $v^{(1)}(x, t) = 0$	at $x = 0 \quad \forall t$
$v(x, t) = 0$ $v^{(2)}(x, t) = 0$	$v^{(2)}(x, t) = 0$ $v^{(3)}(x, t) = 0$	at $x = L \quad \forall t$

The resulting dynamic model is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{F}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{T}_u\mathbf{u} + \mathbf{T}_d\mathbf{d} \tag{2}$$

where $\mathbf{q} = (v_1, \alpha_1, v_2, \alpha_2, \ldots, v_{n+1}, \alpha_{n+1})^T$ is the vector of Lagrangian coordinates, i.e. displacements v_i and slopes α_i at abscissae x_i, $i = 1, \ldots, n+1$ of the spatial discretization (see Figure 1), $\mathbf{M}$ and $\mathbf{K}$ are mass and stiffness matrix respectively, $\mathbf{F}$ is the damping matrix, $\mathbf{u}$ is the vector of control generalized forces, $\mathbf{d}$ is the vector of disturbances acting on the structure and $\mathbf{T}_u$ and $\mathbf{T}_d$ are control and disturbance input matrices respectively. Equation (2) may be rewritten in the equivalent state-space form

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & I \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{F} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 & 0 \\ \mathbf{M}^{-1}\mathbf{T}_u & \mathbf{M}^{-1}\mathbf{T}_d \end{bmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} = \mathbf{A}\mathbf{z} + \mathbf{B}_u\mathbf{u} + \mathbf{B}_d\mathbf{d} \tag{3}$$

where

$$\mathbf{z} = (v_1, \alpha_1, v_2, \alpha_2, \ldots, v_{n+1}, \alpha_{n+1}, \dot{v}_1, \dot{\alpha}_1, \dot{v}_2, \dot{\alpha}_2, \ldots, \dot{v}_{n+1}, \dot{\alpha}_{n+1})^T. \tag{4}$$

3. STATE RECONSTRUCTION

In this section some results of the spline function theory are reported and it is shown how to reconstruct the state variables (4) by means of spline functions.

3.1. SPLINE FUNCTIONS

Let $y(x)$ be a function defined on the interval $x \in [a, b]$ and let $\tilde{\mathbf{y}} = (y_1, \ldots, y_m)$ be its m samples in the correspondence of the ordered knots $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_m)$ with $a = \bar{x}_0 \leqslant \bar{x}_i \leqslant \bar{x}_{m+1} = b$ (see Figure 2).

The cubic spline $\hat{y}(x)$ interpolating these samples is composed of $r = m + 1$ polynomials

$$\beta_{i1}(x - \bar{x}_i)^3 + \beta_{i2}(x - \bar{x}_i)^2 + \beta_{i3}(x - \bar{x}_i) + \beta_{i4} \qquad i = 1, \ldots, r; \qquad x \in [\bar{x}_{i-1}, \bar{x}_i], \tag{5}$$

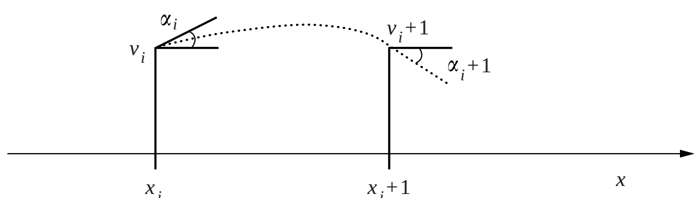

Figure 1. Lagrangian variables.

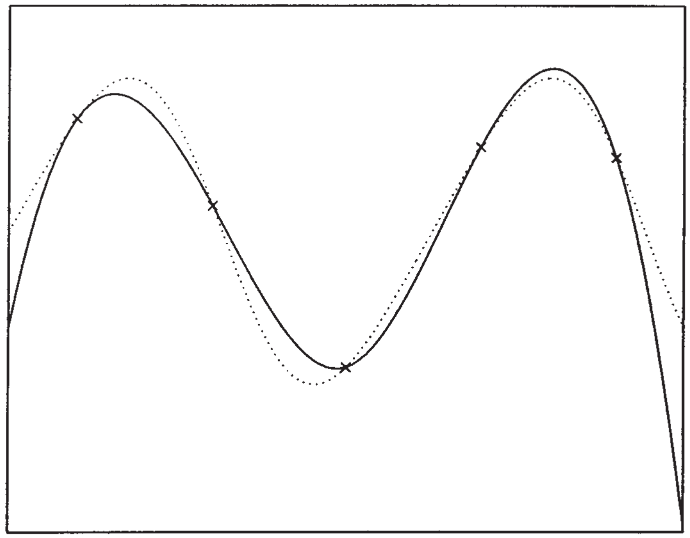

Figure 2. Spline interpolation (data to fit $\times$).

i.e. a cubic polynomial is used to approximate $y(x)$ in each sub-interval $[\bar{x}_{i-1}, \bar{x}_i]$. In the correspondence of the knots the spline fits the samples of $y(x)$, i.e. $\hat{y}(\bar{x}_i) = y(\bar{x}_i)$, $i = 1, \ldots, m$. Moreover, in order to smooth the $\hat{y}(x)$ function, it is imposed that, in the correspondence of knot points, first and second derivative must be continuous.

All these constraints impose $4r - 4$ conditions on the $4r$ parameters β_{ij}, then the user must select conditions on $\hat{y}(x)$ and on its derivatives in the correspondence of the extreme a and b in order to obtain unique solution.

These constraints can be arranged into the following matrix equation

$$\mathbf{D}_0(\bar{\mathbf{x}})\boldsymbol{\beta} = \mathbf{D}_1(\bar{\mathbf{x}})\tilde{\mathbf{y}} \tag{6}$$

where $\boldsymbol{\beta}$ is the vector composed of the $4r$ parameters β_{ij} and the matrices $\mathbf{D}_0$ and $\mathbf{D}_1$ depend on the abscissae $\bar{x}_i$, $i = 0, \ldots, m+1$ and on the conditions imposed by the user on the spline function and its derivatives in a and b [17, 18]. The $\boldsymbol{\beta}$ parameters may be determined from the sample values $\tilde{\mathbf{y}}$ as

$$\boldsymbol{\beta} = \mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\tilde{\mathbf{y}}. \tag{7}$$

The spline interpolation at generic abscissa x can be written as

$$\hat{y}(x) = s(x, \bar{\mathbf{x}})\mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\tilde{\mathbf{y}} \tag{8}$$

where $s(x, \bar{\mathbf{x}})$ is a vector function which selects the piece of cubic spline at abscissa x. From equation (8) it follows that, once the boundary conditions have been specified, the values that the interpolator function $\hat{y}(x)$ assumes in a finite number of specified points $x_1, \ldots, x_n$ can be expressed in the compact form

$$\hat{\mathbf{Y}} = \mathbf{S}(x_1, \ldots, x_n, \bar{\mathbf{x}})\mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\tilde{\mathbf{y}} = \mathbf{T}(x_1, \ldots, x_n, \bar{\mathbf{x}})\tilde{\mathbf{y}} \tag{9}$$

where

$$\mathbf{S}(x_1, \ldots, x_n, \bar{\mathbf{x}}) = \begin{pmatrix} s(x_1, \bar{\mathbf{x}}) \\ \vdots \\ s(x_n, \bar{\mathbf{x}}) \end{pmatrix}. \tag{10}$$

3.1. THE STATE RECONSTRUCTION PROBLEM

Let us return to the beam model described in section 2. The use of spline function to reconstruct the beam deformation may be justified on the basis of the result shown in the Appendix.

The state vector of the spatial discretized model (4) is composed of the following variables:

$$[v(x_1, t), \ldots, v(x_{n+1}, t) \vdots\, \alpha(x_1, t), \ldots, \alpha(x_{n+1}, t) \vdots$$
$$\dot{v}(x_1, t), \ldots, \dot{v}(x_{n+1}, t) \vdots\, \dot{\alpha}(x_1, t), \ldots, \dot{\alpha}(x_{n+1}, t)]^T.$$

Assume that a limited number m of vertical velocity measurements $\dot{\bar{\mathbf{v}}}(t) = (\dot{\bar{v}}_1, \ldots, \dot{\bar{v}}_m)$ at abscissae $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_m)$ are available at each time instant. Moreover, in virtue of Table 1 at the extreme ends of the beam one has the conditions given in Table 2. In other words, the boundary conditions may be seen as further fictitious measurements. The rate of deflection at generic abscissa x and at the points $x_1, \ldots, x_{n+1}$ of the spatial discretization of the beam can be reconstructed via spline function. In particular from equation (8) and (9) one has:

$$\dot{\hat{v}}(x, t) = s(x, \bar{\mathbf{x}})\mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\dot{\bar{v}}(t) \tag{11}$$

$$(\dot{\hat{v}}(x_1, t), \ldots, \dot{\hat{v}}(x_{n+1}, t))^T = \mathbf{T}(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}})\dot{\bar{v}}(t). \tag{12}$$

As far as the remaining state variables are concerned, namely deflection, slope and rate of slope, they can be reconstructed in the following way:

$$(\hat{v}(x_1, t), \ldots, \hat{v}(x_{n+1}, t))^T = \mathbf{T}(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}}) \int_0^t \dot{\bar{v}}(\tau)\, \mathrm{d}\tau \tag{13}$$

where it has been implicitly assumed that $v(x, 0) = 0$. (In the presence of zero-mean inputs, this hypothesis can be relaxed replacing the integrator operator in (14) with a suitable transfer function [17].)

Moreover

$$\dot{\hat{\alpha}}(x, t) = \frac{\mathrm{d}}{\mathrm{d}x} s(x, \bar{\mathbf{x}})\mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\dot{\bar{v}}(t) \tag{14}$$

then

$$\begin{aligned}(\dot{\hat{\alpha}}(x_1, t), \ldots, \dot{\hat{\alpha}}(x_{n+1}, t))^T &= \mathbf{S}_\alpha(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}})\mathbf{D}_0(\bar{\mathbf{x}})^{-1}\mathbf{D}_1(\bar{\mathbf{x}})\dot{\bar{v}}(t) \\ &= \mathbf{T}_\alpha(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}})\dot{\bar{v}}(t)\end{aligned} \tag{15}$$

$$(\hat{\alpha}(x_1, t), \ldots, \hat{\alpha}(x_{n+1}, t))^T = \mathbf{T}_\alpha(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}}) \int_0^t \dot{\bar{v}}(\tau)\, \mathrm{d}\tau \tag{16}$$

TABLE 2

Velocity boundary conditions

Simple supported	Cantilevered	
$\dot{v}(x, t) = 0$ $\dot{v}^{(2)}(x, t) = 0$	$\dot{v}(x, t) = 0$ $\dot{v}^{(1)}(x, t) = 0$	at $x = 0$
$\dot{v}(x, t) = 0$ $\dot{v}^{(2)}(x, t) = 0$	$\dot{v}^{(2)}(x, t) = 0$ $\dot{v}^{(3)}(x, t) = 0$	at $x = L$

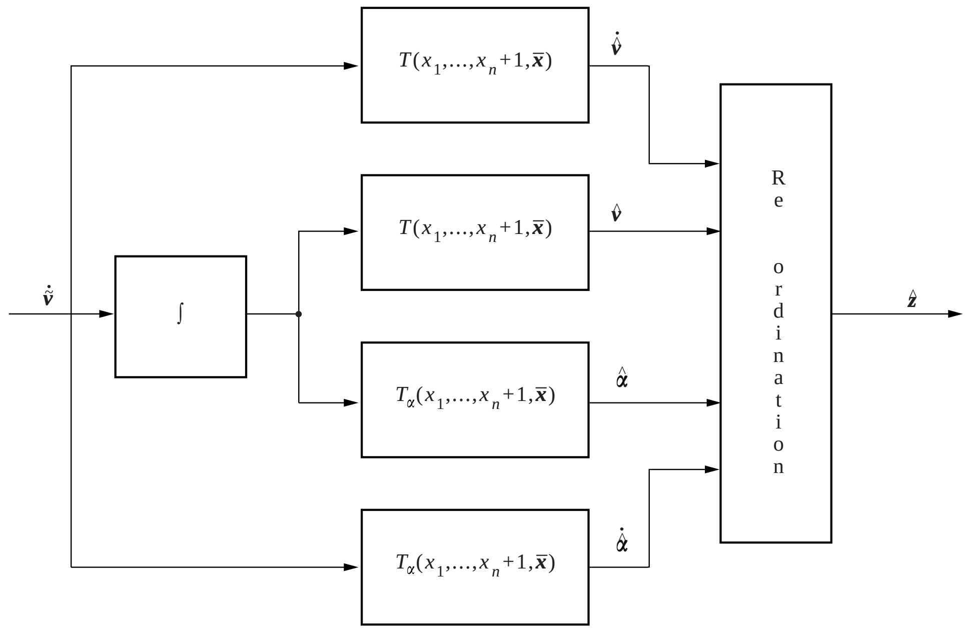

Figure 3. Reconstructor scheme.

where

$$\mathbf{S}_\alpha(x_1, \ldots, x_{n+1}, \bar{\mathbf{x}}) = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} s(x, \bar{\mathbf{x}})|_{x=x_1} \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}x} s(x, \bar{\mathbf{x}})|_{x=x_{n+1}} \end{bmatrix}. \tag{17}$$

The state reconstructor scheme is shown in Figure 3, the *re-ordination* block is a combinatorial one which sorts the reconstructed variables in the same order of the **z** vector. It is interesting to note that the proposed reconstructur is not based on the mathematical model of the beam and it does not use any information about intputs. Then it is insensitive to parameters variations and is able to estimate the state variables, with a bounded error, also in the presence of external persistent disturbances acting on the system through (also unknown) input matrices.

Another interesting feature of the proposed state reconstructor is its intrinsic filtering property. The beam deformation is approximated, at each time instant, by means of a cubic spline. In reference [18] it is shown that spline functions have the property to minimize the overall curvature among all functions fitting the measured values. Then, at each time instant, the deformation is approximated by means of the smoothed spatial frequency modes, i.e. the low frequency modes. This spatial filtering, unlike the classical time-filtering, does not introduce any phase-lag, hence it does not deteriorate stability. A practical consequence of this filtering action is that, even though the inputs excite high-frequency modes, they tend to be screened out in the reconstructed state variables. In section 5 computer simulations show that this spline observer can reduce the spillover phenomenon.

4. FEEDBACK CONTROLLER

In a lot of situations, vibrations in flexible systems are introduced by means of some pseudo-periodic disturbances. In particular, the components of the disturbances close to the natural frequencies of the system (due to the small damping factors of these latter) are highly emphasized. Hence an active vibration controller should be able to reduce the resonance peaks of the system in the correspondence of the bandwidth of the disturbances. Moreover, in order to reduce the spillover phenomenon, the controller should be focused only on these dynamics.

This goal can be reached via an LQ regulator if in the performance index

$$J = \int_0^\infty (\mathbf{z}^T\mathbf{Q}\mathbf{z} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t \tag{18}$$

the matrix $\mathbf{Q}$ is selected as

$$\mathbf{Q} = \boldsymbol{\Gamma}^{-T}\hat{\mathbf{Q}}\boldsymbol{\Gamma}^{-1} \qquad \hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{qI}_{2v} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{19}$$

where $\boldsymbol{\Gamma}$ is the matrix which puts the matrix $\mathbf{A}$ of (3) into the real diagonal form ordered with increasing values of the associated eigenvalues.

Note that minimizing (18) with the weight matrix (19), is equivalent to minimizing

$$\tilde{J} = \int_0^\infty (\boldsymbol{\eta}^T\hat{\mathbf{Q}}\boldsymbol{\eta} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t \tag{20}$$

subject to

$$\dot{\boldsymbol{\eta}} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2 \end{bmatrix}\boldsymbol{\eta} + \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{B}}_2 \end{bmatrix}\mathbf{u} \tag{21}$$

where $\boldsymbol{\eta} = \boldsymbol{\Gamma}^{-1}\mathbf{z}$ are the modal coordinates partitioned in accordance with (19), and $[\tilde{\mathbf{B}}_1^T \;\; \tilde{\mathbf{B}}_2^T]^T = \boldsymbol{\Gamma}^{-1}\mathbf{B}_u$.

Due to exponential stability of system (3), the only semi-definite positive solution $\tilde{\mathbf{P}}$ of the Riccati equation

$$\boldsymbol{\Lambda}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\boldsymbol{\Lambda} - \tilde{\mathbf{P}}\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{B}}\mathbf{R}^{-1}\tilde{\mathbf{B}}^T\boldsymbol{\Gamma}^{-T}\tilde{\mathbf{P}} + \mathbf{Q} = \mathbf{0} \tag{22}$$

has the structure

$$\tilde{\mathbf{P}} = \begin{bmatrix} \tilde{\mathbf{P}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{23}$$

where $\tilde{\mathbf{P}}_1$ is the solution of the Riccati equation

$$\boldsymbol{\Lambda}_1^T\tilde{\mathbf{P}}_1 + \tilde{\mathbf{P}}_1\boldsymbol{\Lambda}_1 - \tilde{\mathbf{P}}_1\tilde{\mathbf{B}}_1\mathbf{R}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}_1 + \mathbf{qI}_{2v} = 0. \tag{24}$$

The optimal control will be

$$\mathbf{u} = \mathbf{R}^{-1}(\mathbf{\Gamma}^{-1}\mathbf{B}_u)^T\tilde{\mathbf{P}}\boldsymbol{\eta} = \mathbf{R}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}_1\boldsymbol{\eta}_1 \tag{25}$$

$$= \mathbf{R}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}_1[\mathbf{I}_{2v} \quad \mathbf{0}]\mathbf{\Gamma}^{-1}\mathbf{z}. \tag{26}$$

It is evident that the control signal depends only on the first v modes. Moreover, the control action modifies exclusively the eigenvalues of the $\mathbf{\Lambda}_1$ block matrix, i.e. only the first v modes are affected by the control signal.

5. SIMULATION RESULTS AND ROBUSTNESS ANALYSIS

Let us consider a beam with the following parameters: $EI = 2{\cdot}55 \times 10^{13}$ [kg · mm^3/s^2], $\mathscr{M} = 10^{-3}$ [kg/mm], $L = 2300$ [mm] with 1% damping factor ξ in each mode, in the two configurations shown in Figure 4.

In order to obtain a finite dimensional model, the beam was divided into 10 elements. Moreover, the following force disturbance, which reproduces the pressure wave recorded on a turbofan aircraft fuselage [17], excites the beams

$$d(t) = \sum_{k=1}^{3} a_k[1 + \sin(\omega_a t)] \sin(k\omega t + d \sin(\omega_p t)) \tag{27}$$

where $\omega = 250$ rad/s, $\omega_a = 25$ rad/s, $\omega_p = 0{\cdot}8$ rad/s, $d = 0{\cdot}1$, $a_1 = 5$ mN, $a_2 = 3$ mN, $a_3 = 2$ mN. The disturbance acts at abscissa $x = 1530$ [mm] of the simple supported beam and at the tip of the span beam.

On the basis of the disturbance spectrum and of the natural frequencies of the systems the matrices $\mathbf{Q}$ and $\mathbf{R}$ in the performance index (18) have been chosen to penalize only the first two modes:

$$\mathbf{Q} = 10^{10}\mathbf{\Gamma}^{-T}\begin{bmatrix}\mathbf{I}_4 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{\Gamma}^{-1}; \qquad \mathbf{R} = \mathbf{I}_3. \tag{28}$$

In particular, the control signal used in the feedback scheme is

$$\mathbf{u} = \mathbf{R}^{-1}\mathbf{B}_u^T\mathbf{P}\hat{\mathbf{z}} \tag{29}$$

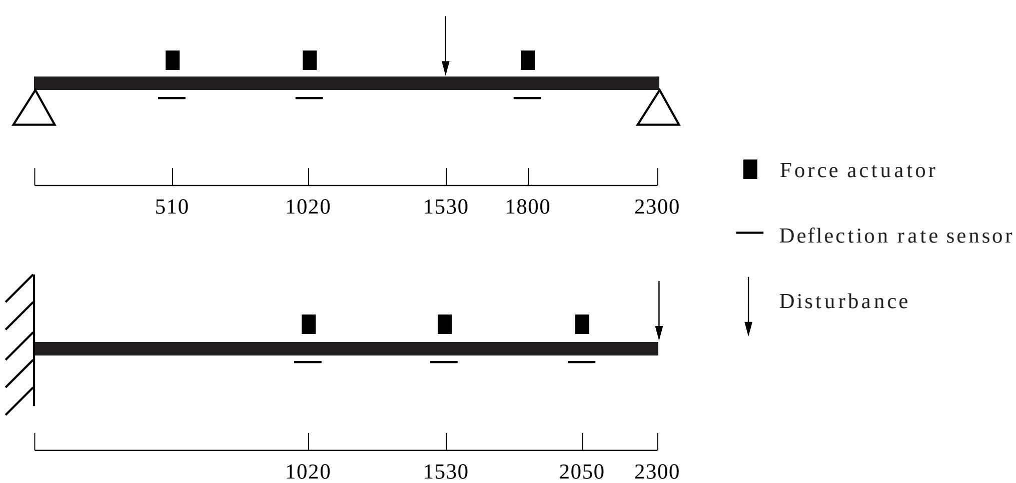

Figure 4. Beam configurations.

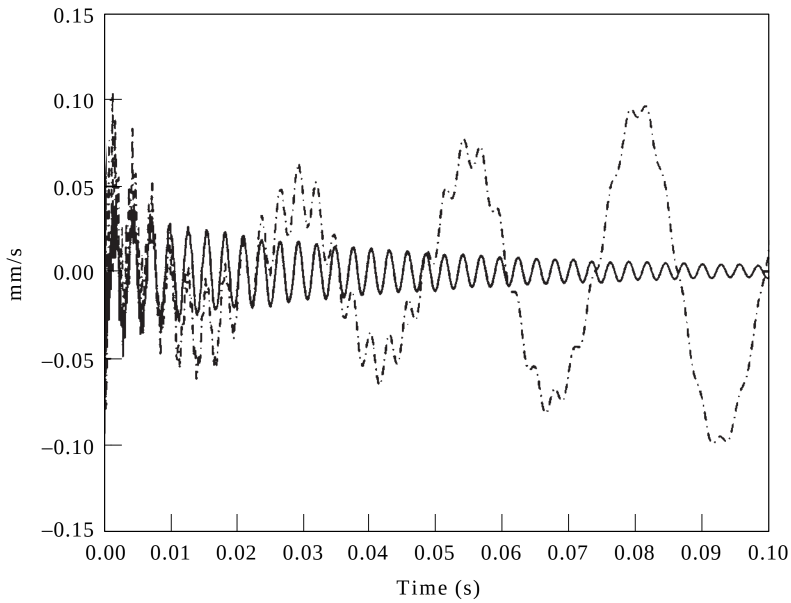

Figure 5. Time history of the simple supported beam at $x_a = 1020$ (open loop, dashed line; optimal feedback with spline reconstructor, solid line).

where $\hat{\mathbf{z}}$ is the spline reconstruction of the state variables $\mathbf{z}$ obtained from the three velocity deflection sensors and using the reconstruction scheme of Figure 3. In order to reduce the influence of the initial estimation error, the integrator blocks of Figure 3 have been replaced by transfer functions $1/(s + 10^{-6})$.

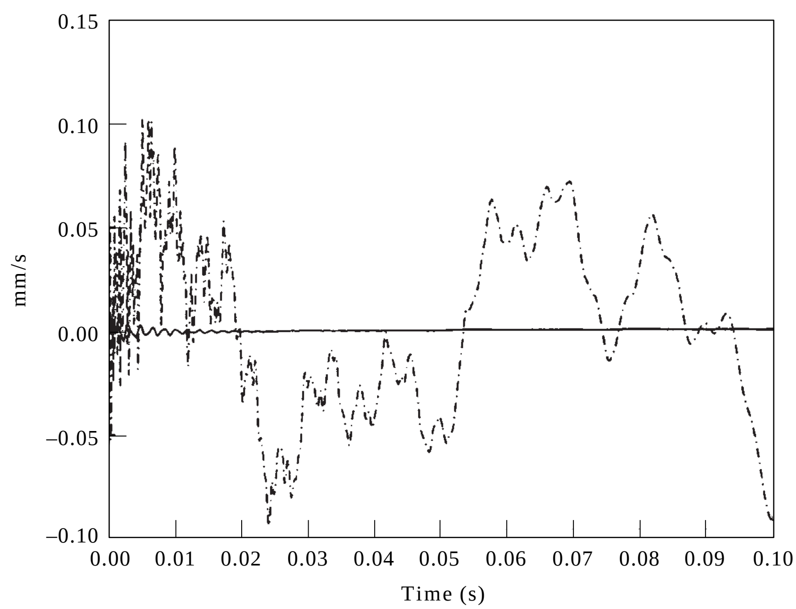

Figure 6. Time history of the cantilever beam at $x_b = 1530$ (open loop, dashed line; optimal feedback with spline reconstructor, solid line).

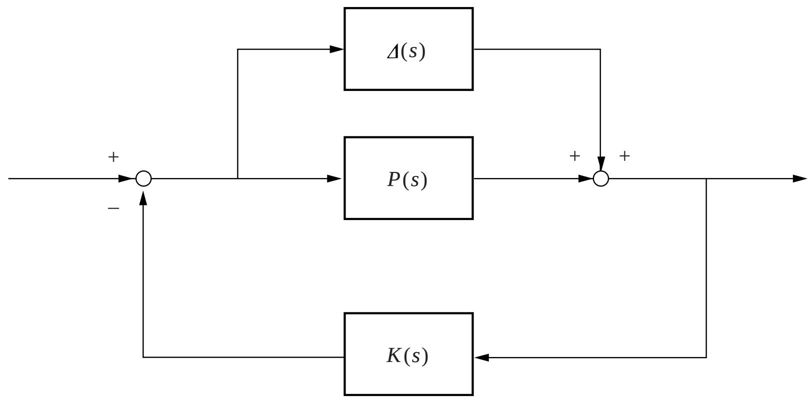

Figure 7. Additive uncertainties.

The performance of the proposed controller (optimal feedback of the reconstructed state) is shown in Figure 5 where open loop and closed loop behaviours of the simple supported beam at abscissa $x_a = 1020$ are compared. The system is forced by the disturbance (27) from the instant $t = 0$ and with the following initial condition

$$\mathbf{z}(0) = 0{\cdot}25\mathbf{B}_d. \tag{30}$$

It is evident that in the steady-state the controller is able to reduce the vibrations. In the transient, when the output largely depends on the high-frequency dynamics, open loop and closed loop behaviours are similar. The performance obtained with (29) is almost

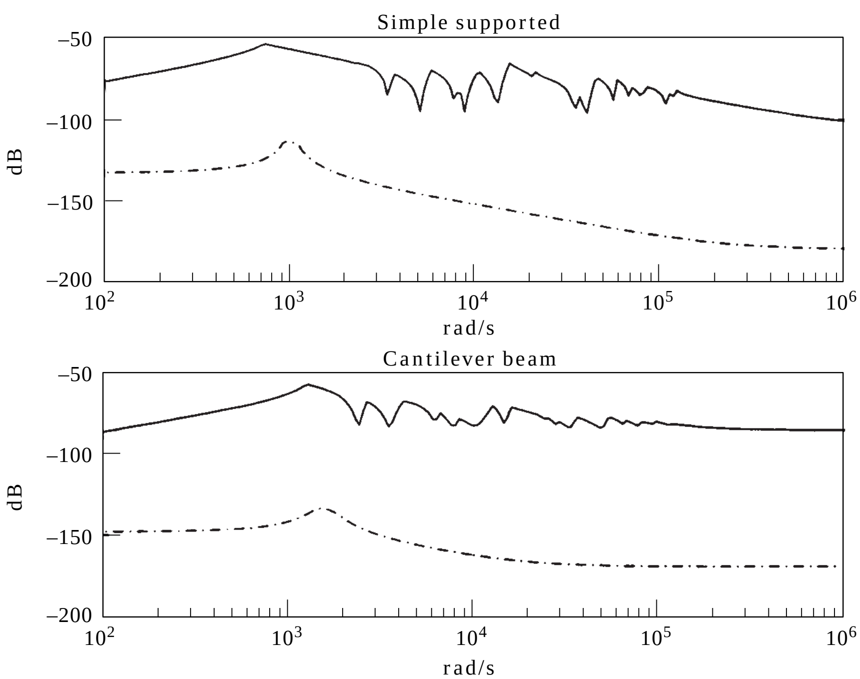

Figure 8. Diagram of the robust stability margin function (optimal feedback, dashed line; optimal feedback with spline reconstructor, solid line).

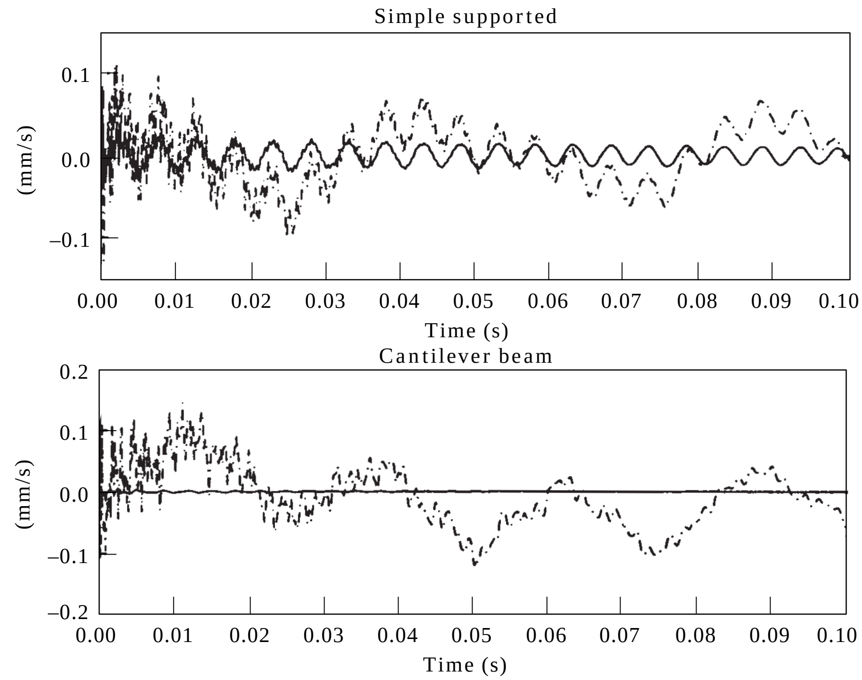

Figure 9. Time histories at abscissae x_a and x_b, respectively, of the perturbed more detailed models in the presence of a controller designed on the "10-part model" and applied by means of the spline reconstructor (open loop, dashed line; optimal feedback with spline reconstructor, solid line).

indistinguishable to that obtained using direct feedback of the state variables (for this reason this latter has not been reported in the figures).

The results for the cantilever beam have been shown in Figure 6. In this case the action of the controller is more efficient.

In order to evaluate the robustness of the control scheme the Small Gain Theorem has been used. Consider now the block diagram of Figure 7 where $\Delta(s)$ represents a stable perturbation from a nominal value $P(s)$. The parameter $\Delta(s)$ takes into account both the plant-parameters uncertainties and the neglected dynamics. In reference [4] it is shown that,

TABLE 3

Beam characteristic functions

	Characteristic equation	$V_i(x)$	σ_i
Simple supported	$\sin\lambda = 0 \Rightarrow \lambda_i = i\pi$	$\sin\frac{\lambda_i x}{L}$	
Cantilever beam	$\cosh\lambda\cos\lambda + 1 = 0$	$\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left(\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right)$	$\sigma_i = \frac{\cosh\lambda_i + \cos\lambda_i}{\sinh\lambda_i + \sin\lambda_i}$

also for an infinite dimensional system, a controller $K(s)$ which stabilizes $P(s)$ stabilizes all the perturbed system such that

$$\bar{\sigma}(\Delta(j\omega)) \leqslant \frac{1}{\bar{\sigma}(K(j\omega)(I + K(j\omega)P(j\omega))^{-1})} \tag{31}$$

where $\bar{\sigma}$ represents the maximum singular value. This equation gives a bound on the maximum amplitude of the perturbation which does not destabilize the closed loop system. Then the right side of this equation can be considered as a robust stability margin function (RSMF).

In Figure 8 the RSMFs obtained for the direct state feedback and using the spline reconstructor are compared. The presence of the spline reconstructor greatly increases the robustness of the scheme. Note that at very low frequencies this is not true and the spline-RSMF drops below to the direct state feedback-RSMF; however, generally, at low frequencies model errors are negligible.

To have a confirmation of the stability robustness property, the previously designed controller has been tested on a "more accurate" model obtained by dividing each beam element into two parts. Moreover, it has been assumed the following parameter errors

$$\Delta EI = -30\% \qquad \Delta\xi = -15\%. \tag{32}$$

As shown in Figure 9 there are no performance degradation in the closed loop response, even the "parameter errors" have deeply modified the open loop behaviour. On the other side, the direct feedback of state variables associated with the "10 part-model" drives the system to instability.

6. CONCLUSION

In this paper an active vibration controller for a flexible beam has been presented. The controller is based on an LQ regulator and it uses a spline approach to reconstruct the state variables from sensors measurements. The spline reconstructor is able to reconstruct, with a bounded error, the first v modes of the structure and only these modes are weighted into the LQ performance index. This allows one to focus the control action only on the prescribed dynamics. Moreover, the spline reconstructor introduces a sort of spatial filtering on the un-modelled dynamics which allows one to reduce the spillover phenomenon.

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APPENDIX

In this section it is shown that a spline function is able to reconstruct, with a bounded error, the free vibration of a beam like structure. It is well known [19] that the solution of (1) is

$$v(x,t)=\sum_{i=1}^{\infty} V_i(x)\sin\left(\frac{\lambda_i^2}{L^2}\sqrt{\frac{EI}{\mathscr{M}}}\,t+\phi_i\right) \tag{33}$$

where λ_i are the solutions of the characteristic equation, and $V_i(x)$ are the eigenfunctions. The values assumed by λ_i and $V_i(x)$ depend on the boundary conditions; as shown in Table 3. Before presenting the main result of the spline reconstructor, one needs to recall the following result on spline functions

Lemma 1([18])

If a function $y(x)$, *defined on the interval* $x\in[a,b]$, *has a continuous fourth derivative then the error between* $y(x)$ *and the cubic spline* $\hat{y}(x)$ *fitting the data* $(\bar{x}_i, y(\bar{x}_i))$ *is bounded by*

$$\|y-\hat{y}\|\leqslant \tfrac{5}{384}\Delta^4\|y^{(4)}\| \tag{34}$$

where Δ *is the maximum distance between two consecutive knots and*

$$\|y\|\triangleq\max_{x\in[a,b]}|y(x)| \tag{35}$$

Theorem 1. Consider the free vibration $v(x,t)$ of a beam in the presence of m discrete measurement points. Assume that the initial conditions excite at the most the first ν modes of the beam. Say $\hat{v}^s(x,t)$ the cubic spline which fits, at each time instant, the available measurements and satisfies at extreme ends conditions analogous to the boundary conditions of the beam. Then the approximation error is bounded, and an upper bound on the error is given by

$$\|v - \hat{v}^s\| \leqslant \frac{5}{384}\frac{\Delta^4}{L^4} V_{\max}\Phi(v) \qquad \forall t \tag{36}$$

where Δ is the maximum distance between two consecutive sensors (considering the beam extreme as further fictitious sensors), $V_{\max} \triangleq \max_{i,x}\|V_i(x)\|$, $i = 1, \ldots, v$, $x \in [0, L]$ is the maximum amplitude of the excited modes and

$$\Phi(v) = \lambda_1^4 + \lambda_2^4 + \cdots + \lambda_v^4. \tag{37}$$

Proof. The hypothesis on initial conditions imply that the summation in (33) is limited to the first v terms. Moreover

$$\ddot{v}(x, t) = -\sum_{i=1}^{v}\left(\frac{\lambda_i}{L}\right)^4 \frac{EI}{\mathscr{M}} V_i(x) \sin\left(\frac{\lambda_i^2}{L^2}\sqrt{\frac{EI}{\mathscr{M}}}\, t + \phi_i\right) \tag{38}$$

then

$$\|\ddot{v}\| \leqslant \frac{1}{L^4}\frac{EI}{\mathscr{M}}(\lambda_1^4\|V_1\| + \lambda_2^4\|V_2\| + \cdots + \lambda_v^4\|V_v\|)$$

$$\leqslant \frac{1}{L^4}\frac{EI}{\mathscr{M}} V_{\max}(\lambda_1^4 + \lambda_2^4 + \cdots + \lambda_v^4) = \frac{1}{L^4}\frac{EI}{\mathscr{M}} V_{\max}\Phi(v) \tag{39}$$

Moreover, from (1)

$$\|v^{(4)}\| \leqslant \frac{\mathscr{M}}{EI}\|\ddot{v}\| \leqslant \frac{1}{L^4} V_{\max}\Phi(v) \tag{40}$$

and, using (34) of Lemma 1, we have the assertion. ■

Note that, for the simple supported configuration the function $\Phi(v)$ is bounded by

$$\Phi(v) \leqslant \frac{(v+1)^5 - 1}{5}\pi^4. \tag{41}$$

NOMENCLATURE

$v(x, t)$	beam deformation
$v^{(i)}$	ith derivative with respect to spatial variable
$\dot{v}, \ddot{v}$	first and second derivative with respect to time
n	number of element of the discretized beam model
L	beam length
$x_1, \ldots, x_n$	beam abscissae of the spatial discretization
$v_i, \dot{v}_i$	deflection and deflection rate at abscissa x_i
$\alpha_i, \dot{\alpha}_i$	slope and slope rate at abscissa x_i
$\|v(x, t)\|$	maximum of $\lvert v(x, t)\rvert$ with respect to x in $[0, L]$
Δ	maximum distance between two knots of the spline function
m	number of available measurements
$\bar{\mathbf{x}} = [\bar{x}_1, \ldots, \bar{x}_m]$	abscissae of the measurements
$\tilde{\dot{\mathbf{v}}} = (\tilde{\dot{v}}_1, \ldots, \tilde{\dot{v}}_m)$	vertical velocity measurements
$\hat{v}, \hat{\alpha}, \hat{\dot{v}}, \hat{\dot{\alpha}}$	spline reconstruction of the discretized beam quantities
$\hat{y}, \hat{v}^s$	spline reconstruction of $y(x)$ and $v(x, t)$ respectively
$\bar{\sigma}(F)$	maximum singular value of F
$\mathbf{I}_n$	identity matrix of order n.